Department of CIS
Subject: Algorithms
Total Marks: 35

Task 1: 10 Marks

a) [5 marks] For each of the statements below, state whether it is true or false and then prove your answer.

(a) $15n^3 \log n + 10n^2 + 50$ is $O(n^3 \log n)$.
(b) $3n^2 - 12n + 2$ is $\Omega(n^3)$
(c) 2^{n+1} is $\Theta(2^n)$
(d) 2^{2n} is $O(2^n)$
(e) $\log(n!)$ is $O(n \log n)$ (Hint: compare $n!$ and n^n)

b) [5 marks] There are some situations where we are asked to sort data that is almost sorted. A k-sorted array contains no element that is more than k positions from its position in the properly sorted array. For the questions below, A is a k-sorted array with k << n; What are the runtimes of Insertion-Sort, Merge-Sort and Quicksort on A? Explain your answers.

Task 2: 15 Marks

a) In a binary search tree, we might also keep track of the total number of nodes in that subtree (including the node itself). Assuming we store this value (e.g. x.size).

i. [5 marks] write pseudocode for a function BSTKeyLessThan(T, k) that takes a tree T and a number k and returns the number of values in the tree T that are less than k. For example, if the tree had the number 1 through 9 in it, then BSTKeyLessThan(T, 5) should return 4.

ii. [4 marks] What is the best-case and worst-case running time of your algorithm?

b) Given an undirected graph G with nonnegative edge weights $W \geq 0$. Suppose you have calculated the minimum spanning tree of G and also the shortest paths to all nodes from a particular node $s \in V$. Now, suppose that each edge weight is increased by 1, i.e. the new weights $W_n = W + 1$.

i. [3 marks] Does the minimum spanning tree change? Give an example where it does or prove that it cannot change.

ii. [3 marks] Do the shortest paths from s change? Given an example where it does or prove that it cannot change.

Task 3: 10 Marks

A thief robbing a bulk food store finds n items worth $v_1, v_2, \ldots v_n$ dollars and weigh $w1, w2, , wn$ pounds, where vi and wi are integers. The thief can carry at most W pounds in the knapsack. Because it's a bulk food store, the thief may take all of an item or only some fraction of any item (e.g. half of item i, getting value vi/2 with weight wi/2). The goal of the thief is to select the items so as to maximize the profit while staying under the weight constraint.

(a) [5 marks] Describe an optimal greedy heuristic and a high-level algorithm for selecting which items (and how much) to select. You don't need to state the low-level details of the algorithm, just how you will construct your solution.

(b) [5 marks] Prove that your algorithm is correct. You may either use a "stays ahead" proof or a proof by contradiction.